Mid-Term Examination
ALGEBRA-III
27th September 2002

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1. (a) Define a primitive element in $Z[X]$. If a primitive element divides another element of $Z[X]$ in $Q[X]$, show that the division takes place in $Z[X]$ itself. (7)

   (b) Let p be a prime number. Prove that $X^{p-1} + X^{p-2} + \cdots + X + 1$ is irreducible in $Z[X]$. (3)

2. (a) Determine all polynomials f in $F_q[X]$ such that $f(\alpha) = 0$ for all $\alpha \in F_q \cancel{[X]}$. (4)

   (b) Factor $X^9 - X$ and $X^{27} - X$ over F_3 in to irreducible polynomials. (6)

3. (a) Let F be a field and $f \in F[X]$. Show that there exists a finite extension L of F such that f factors into linear polynomials in $L[X]$. (6)

   (b) Determine the degree of the splitting field of the following polynomials over Q:
   $X^4 - 1, \quad X^3 - 2, \quad X^4 + 1.$ (4)

4. Let K be a splitting field of the polynomial $X^4 - 3$ over Q.

   (a) Prove that $[K : Q] = 8$. (3)

   (b) Prove that the Galois group of K over Q is dihedral and describe the action of this group on the roots of the above polynomial. (7)

5. (a) Let $F \subset L \subset K$ be fields. Prove or disprove:
   1. If K/F is Galois, then K/L is also Galois.
   2. If K/F is Galois, then L/F is also Galois.
   3. If L/F and K/L are Galois, then so is K/F. (7)

   (b) Determine all automorphisms of the field $Q(\sqrt[3]{5})$. (3)

6. (a) Let K/F be a Galois extension whose Galois group is S_3. Prove or disprove: K is the splitting field of an irreducible cubic equation over F. (6)

   (b) Prove the best generalisation you can of the above problem. (4)

7. (a) Let $F(\alpha)$ be a quadratic extension of F such that $\alpha^2 \in F$. Determine all the elements of $F(\alpha)$ whose sqaures are in F. (4)

   (b) Let β be the real root of $X^3 + X + 1$. Prove or disprove: $Q(\beta)$ is generated by a cube root of a rational number. Compare your result with Kummer's theorem on prime degree extensions of certain fields. (6)